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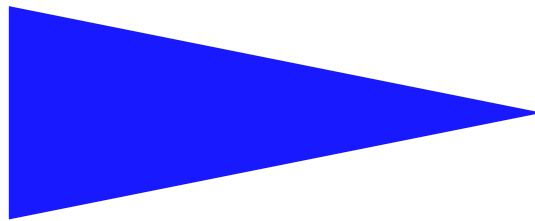
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STRUCTURAL PRESBURGER-DEFINABLE  
DIGIT VECTOR AUTOMATA

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## Structural Presburger-definable Digit Vector Automata

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**Abstract:** Digit Vector Automata (DVA) provide a natural symbolic representation for regular sets of integer vectors encoded as strings of digit vectors (least significant digit first). We prove that the minimal DVA that represents a Presburger-definable set is structurally Presburger-definable: that means, the DVA obtained by modifying the initial state and the set of final states represents a Presburger-definable set.

**Key-words:** Automata, Presburger arithmetic, Semi-linear set, Symbolic representation

*(Résumé : tsvp)*



# Structure des automates Presburger-définissables

**Résumé :** Les automates finis permettent de représenter symboliquement des ensembles infinis de vecteurs d'entiers, décomposés comme des mots de vecteurs de chiffres. On montre que l'automate minimal représentant un ensemble Presburger-définissable, est structurellement Presburger-définissable: c'est à dire, que les automates obtenus en changeant l'état initial et les états finaux représentent des ensembles Presburger-définissables.

**Mots clés :** Automate, Arithmétique de Presburger, Ensemble semilinéaire, Représentation symbolique

Presburger arithmetic [21] is a decidable logic used in a large range of applications. Different techniques [11] and tools have been developed for manipulating *the Presburger-definable sets* (the sets of integer vectors satisfying a Presburger formula): by working directly on the Presburger-formulas (implemented in OMEGA [20]), by using semi-linear sets [12] (implemented in BRAIN [22]), or by using Digit Vector Automata (DVA) that represent regular sets of integer vectors encoded as strings of digit vectors, least or most significant digit first [23, 7] (implemented in FAST [1], LASH [15] and CSL-ALV [2]). Presburger-formulas and semi-linear sets lack canonicity: there does not exist a natural way to canonically represent a set. As a direct consequence, a set that possesses a simple representation could unfortunately be represented in an unduly complicated way. Moreover, deciding if a given vector of integers is in a given set, is at least *NP-hard* [4, 12]. On the other hand, a minimization procedure for automata provides a canonical representation for *DVA-definable sets* (a set represented by a DVA). That means, the DVA that represents a given set only depends on the set and not on the way we have computed it. For this reason, DVA are well adapted for applications that require a lot of Boolean manipulations like model-checking.

Recently, the DVA obtained by modifying the set of final states, has provided some applications. First, we have proved that modifying the set of final states of a DVA, provides some simple sets that can be used for deciding in polynomial time if a DVA is Presburger-definable (that means, the DVA represents a Presburger-definable set) [17]. Recall that the previous algorithm for deciding this property, was given by Muchnik in 1991 [18, 19, 8], and works in *quadruply-exponential time*. Second, Bartzis and Bultan [3] provided a *widening operator* for DVA in order to enforce the convergence of the incrementally computed DVA, during the reachability state space exploration of an *infinite state system*. This operator is obtained by modifying the set of final states of Presburger-definable DVA, but they do not prove that the obtained DVA remain Presburger-definable.

However, from practical and theoretical point of view, working only with Presburger-definable DVA has some advantages. First the manipulation complexity (boolean operations and variable elimination) is at most 3-exponential time for Presburger-definable DVA (see [13, 17]) and non-elementary for general DVA (see [5]). Second, we can compute in polynomial time, a Presburger-formula that defines the set represented by a Presburger-definable DVA. Then this formula can be used in other tools like OMEGA.

In this paper, we introduce a new automata-based representation for regular subsets of  $\mathbb{Z}^m$ , called the *digit Vector automata (DVA)*. Even if DVA are very similar to other automata-based representations [6, 7, 8], it is the *first* automata-based representation for any regular subsets of  $\mathbb{Z}^m$ , that is both *canonical* (there exists a unique minimal DVA that represents a given set  $X$ ) and *stable by modifying the initial state* (this stability provides a natural way for associating a subset of  $\mathbb{Z}^m$  to any state of the DVA). Moreover, we prove that the minimal DVA that represents a Presburger-definable set is structurally Presburger-definable: that means, any DVA obtained by modifying the initial state and the set of final states, is Presburger-definable.

## 1 Notations

We denote by  $\mathbb{Z}$  and  $\mathbb{N} \setminus \{0\}$  respectively the set of integers and non-negative integers. The set  $X^m$  is called the set of vectors with  $m \in \mathbb{N}$  components in a set  $X$ . Given an integer  $i \in \{1, \dots, m\}$  and a vector  $x \in X^m$ , the  $i$ -th component of  $x$  is written  $x[i] \in X$ . We denote by  $\mathbf{e}_0$  the vector  $\mathbf{e}_0 = (0, \dots, 0)$ . Vectors  $x + y$  and  $t.x$  are defined by  $(x + y)[i] = (x[i] + y[i])$  and  $(t.x)[i] = t.(x[i])$  for any  $i \in \{1, \dots, m\}$ ,  $x, y \in \mathbb{Q}^m$ ,  $t \in \mathbb{Q}$ . We denote by  $\langle x, y \rangle = \sum_{i=1}^m x[i].y[i]$ , the *dot product* of two vectors  $x, y \in \mathbb{Q}^m$ . Given a *functions*  $f : X \rightarrow Y$ ,  $A \subseteq X$  and  $B \subseteq Y$ , we define  $f(A) = \{f(a); a \in A\}$  and  $f^{-1}(B) = \{x \in X; f(x) \in B\}$ .

Given a non-empty finite *alphabet*  $\Sigma$ , we denote by  $\Sigma^+$  the set of non-empty *words* over  $\Sigma$  and we denote by  $\epsilon$  the empty word. As usual  $\Sigma^*$  denotes the set of words  $\Sigma^+ \cup \{\epsilon\}$ . A subset  $\mathcal{L} \subseteq \Sigma^*$  is called a *language*. The concatenation of two words  $\sigma_1$  and  $\sigma_2$  (resp. two languages  $\mathcal{L}_1$  and  $\mathcal{L}_2$ ) is denoted by  $\sigma_1.\sigma_2$  (resp.  $\mathcal{L}_1.\mathcal{L}_2 = \{\sigma_1.\sigma_2; (\sigma_1, \sigma_2) \in \mathcal{L}_1 \times \mathcal{L}_2\}$ ). Given a word  $\sigma \in \Sigma^*$ , we denote by  $(\sigma^i)_{i \in \mathbb{N}}$  the

sequence of words defined by the induction  $\sigma^0 = \epsilon$  and  $\sigma^{i+1} = \sigma^i \cdot \sigma$ . We denote by  $\sigma^*$  the language  $\sigma^* = \{\sigma^i; i \in \mathbb{N}\}$ . The *length* of a word  $\sigma$  is denoted by  $|\sigma| \in \mathbb{N}$ . For any non-empty word  $\sigma \in \Sigma^+$ , we denote by  $\sigma[1], \dots, \sigma[|\sigma|]$  the elements in  $\Sigma$  such that  $\sigma = \sigma[1] \dots \sigma[|\sigma|]$ .

## 2 Digit Vector Automata

In this section, the *Digit Vector Automata (DVA)* representation, a state-based representation of set of integer vectors, is presented. The sets obtained by *moving the initial state* and *modifying the set of final states* of a DVA are respectively characterized in sections 2.2 and 2.3.

### 2.1 Digit vector decomposition

Let us consider an integer  $r \geq 2$  called the *basis of decomposition* and the *set of digits*  $\Sigma_r = \{0, \dots, r-1\}$ . In this section, we study the *least significant digit first decomposition* of an integer vector in  $\mathbb{Z}^m$  into a word of *digit vectors* in  $(\Sigma_r^m)^*$ . This decomposition can be easily obtained by considering the sequence  $(\gamma_{r,\sigma})_{\sigma \in (\Sigma_r^m)^*}$  of functions  $\gamma_{r,\sigma} : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$  uniquely defined by the following equalities [16]:

$$\begin{cases} \gamma_{r,b}(x) = r \cdot x + b & (b, x) \in \Sigma_r^m \times \mathbb{Z}^m \\ \gamma_{r,\sigma_1 \cdot \sigma_2} = \gamma_{r,\sigma_1} \circ \gamma_{r,\sigma_2} & (\sigma_1, \sigma_2) \in (\Sigma_r^m)^* \times (\Sigma_r^m)^* \end{cases}$$

Assume that the dimension  $m$  is equal to 1 and consider a couple  $(\sigma, s) \in \Sigma_r^* \times S_r$  where  $S_r$  is the set of *sign digits*  $S_r = \{0, r-1\}$ . The following equality is called the *least significant digit first decomposition with 2-complement*:

$$\gamma_{r,\sigma} \left( \frac{s}{1-r} \right) = \begin{cases} \sum_{i=1}^{|\sigma|} r^{i-1} \sigma[i] \in \mathbb{N} & \text{if } s = 0 \\ \sum_{i=1}^{|\sigma|} r^{i-1} \sigma[i] - r^{|\sigma|} \in \mathbb{Z} \setminus \mathbb{N} & \text{if } s = r-1 \end{cases}$$

The previous decomposition shows intuitively that  $s = 0$  correspond to the *non-negative sign digit* whereas  $s = r-1$  corresponds to the *negative one*.

For a general dimension  $m \geq 1$ , let us consider the function  $\rho_r : (\Sigma_r^m)^* \times S_r^m \rightarrow \mathbb{Z}^m$  defined by the following equality:

$$\rho_r(\sigma, s) = \gamma_{r,\sigma} \left( \frac{s}{1-r} \right)$$

A couple  $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$  such that  $x = \rho_r(\sigma, s)$  is called a *r-decomposition* of  $x \in \mathbb{Z}^m$ . Remark that any  $x \in \mathbb{Z}^m$  owns at least one *r-decomposition*.

Function  $\rho_r$  naturally associate to any language  $\mathcal{L} \subseteq (\Sigma_r^m)^* \times S_r^m$  a subset  $X = \rho_r(\mathcal{L})$  of  $\mathbb{Z}^m$ . Remark however that there exists some languages  $\mathcal{L}_1, \mathcal{L}_2$  and  $\mathcal{L}$  such that  $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}$  and such that  $\rho_r(\mathcal{L}_1) \cap \rho_r(\mathcal{L}_2) \neq \rho_r(\mathcal{L})$ . For instance, consider  $\mathcal{L}_1 = \{(\epsilon, 0)\}$ ,  $\mathcal{L}_2 = \{(0, 0)\}$  and  $\mathcal{L} = \emptyset$ . Such a side effect is due to the fact that an integer vector  $x \in \mathbb{Z}^m$  does not have a unique *r-decomposition*. The following lemma characterizes *r-decompositions* associated to the same vector.

**Lemma 1** *Two r-decompositions  $(\sigma_1, s_1)$  and  $(\sigma_2, s_2)$  are associated to the same vector if and only if  $s_1 = s_2$  and  $\sigma_1 \cdot s_1^* \cap \sigma_2 \cdot s_2^* \neq \emptyset$ .*

**Proof :** Let us first remark that for any sign digit vector  $s \in S_r^m$ , we have  $\gamma_{r,s}(\frac{s}{1-r}) = \frac{s}{1-r}$ . In particular, we have  $\rho_r(\sigma \cdot s^k, s) = \rho_r(\sigma, s)$  for any word  $\sigma \in (\Sigma_r^m)^*$  and for any  $k \in \mathbb{N}$ . This equality is well known when  $s = 0$  and it just means that *adding extra zero digits* to the least significant digit first decomposition of a non-negative integer does not change its value.

Assume first that  $(\sigma_1, s_1)$  and  $(\sigma_2, s_2)$  are such that  $s_1 = s_2$  and  $\sigma_1 \cdot s_1^* \cap \sigma_2 \cdot s_2^* \neq \emptyset$ , and let us prove that  $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$ . There exist  $k_1, k_2 \in \mathbb{N}$  such that  $\sigma_1 \cdot s_1^{k_1} = \sigma_2 \cdot s_2^{k_2}$ . In particular, from the previous paragraph we deduce  $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_1 \cdot s_1^{k_1}, s_1) = \rho_r(\sigma_2 \cdot s_2^{k_2}, s_2) = \rho_r(\sigma_2, s_2)$ .

Next, assume that  $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$  and let us prove that  $s_1 = s_2$  and  $\sigma_1.s_1^* \cap \sigma_2.s_2^* \neq \emptyset$ . As the manipulated structures are defined component wise, we can assume without loss of generality that the dimension  $m$  is equal to 1. Remark that the sign digits  $s_1$  and  $s_2$  must be equal. In fact, otherwise, there exists  $i_1, i_2 \in \{1, 2\}$  such that  $s_{i_1} = 0$  and  $s_{i_2} = r - 1$  and in this case we have shown that  $\rho_r(\sigma_{i_1}, s_{i_1}) \in \mathbb{N}$  and  $\rho_r(\sigma_{i_2}, s_{i_2}) \in \mathbb{Z} \setminus \mathbb{N}$  which is in contradiction with  $\rho_r(\sigma_1, s_1) = \rho_r(\sigma_2, s_2)$ . Let us consider  $k_1, k_2 \in \mathbb{N}$  such that the words  $w_1 = \sigma_1.s_1^{k_1}$  and  $w_2 = \sigma_2.s_2^{k_2}$  have the same length denoted by  $k \in \mathbb{N}$ . The first paragraph shows that  $\rho_r(w_1, s_1) = \rho_r(w_2, s_2)$ . As  $s_1 = s_2$ , we deduce the following equality:

$$\sum_{i=1}^k r^{i-1} \cdot (w_1[i] - w_2[i]) = 0$$

Assume by contradiction that  $w_1 \neq w_2$ . In this case  $k \in \mathbb{N} \setminus \{0\}$  and there exists a maximal (for  $\leq$ )  $j \in \{1, \dots, k\}$  such that  $w_1[j] \neq w_2[j]$ . We have:

$$|w_1[i] - w_2[i]| \begin{cases} = 0 & \text{if } i > j \\ \geq 1 & \text{if } i = j \\ \leq r - 1 & \text{if } i < j \end{cases}$$

We deduce the following bound::

$$\begin{aligned} \left| \sum_{i=1}^k r^{i-1} \cdot (w_1[i] - w_2[i]) \right| &= |r^j \cdot (w_1[j] - w_2[j]) + \sum_{i=1}^{j-1} r^{i-1} \cdot (w_1[i] - w_2[i])| \\ &\geq |r^j \cdot (w_1[j] - w_2[j])| - \sum_{i=1}^{j-1} |r^{i-1} \cdot (w_1[i] - w_2[i])| \\ &\geq r^j - \sum_{i=1}^{j-1} r^{i-1} \cdot (r - 1) \\ &= 1 \end{aligned}$$

We obtain a contradiction. We deduce that  $w_1 = w_2$  and in particular the word  $w = w_1 = w_2$  is in  $\sigma_1.s_1^* \cap \sigma_2.s_2^*$ . Q.E.D

A language  $\mathcal{L} \subseteq (\Sigma_r^m)^* \times S_r^m$  is said *saturated* [14] if for any  $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$ , we have  $(\sigma, s) \in \mathcal{L}$  if and only if  $(\sigma.s, s) \in \mathcal{L}$ . Previous lemma 1 shows that a language  $\mathcal{L}$  is saturated if and only if there exists  $X \subseteq \mathbb{Z}^m$  such that  $\mathcal{L} = \rho_r^{-1}(X)$ . In particular, we deduce that the *side effect*  $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}$  and  $\rho_r(\mathcal{L}_1) \cap \rho_r(\mathcal{L}_2) \neq \rho_r(\mathcal{L})$  is no longer true for saturated language. In fact, for any saturated languages  $\mathcal{L}_1, \mathcal{L}_2$  and for any  $\# \in \{\cup, \cap, \setminus, \Delta\}$ , the language  $\mathcal{L}_1 \# \mathcal{L}_2$  is saturated and  $\rho_r(\mathcal{L}_1) \# \rho_r(\mathcal{L}_2) = \rho_r(\mathcal{L}_1 \# \mathcal{L}_2)$ .

We are interested in associating to a saturated language a *state-based symbolic representation*, called *Digit Vector Automata*.

**Definition 1 (Digit Vector Automata)** A Digit Vector Automaton (DVA)  $\mathcal{A}$  is a tuple  $\mathcal{A} = (Q, \Sigma_r^m, \delta, q_0, F_0)$  where:

- $Q$  is a non-empty finite set of states.
- $\delta : Q \times \Sigma_r^m \rightarrow Q$  is the transition function.
- $q_0 \in Q$  is the initial state.
- $F_0 \subseteq Q \times S_r^m$  is the set of final states such that  $(q, s) \in F_0$  if and only if  $(q', s) \in F_0$  for every  $q' = \delta(q, s)$ .



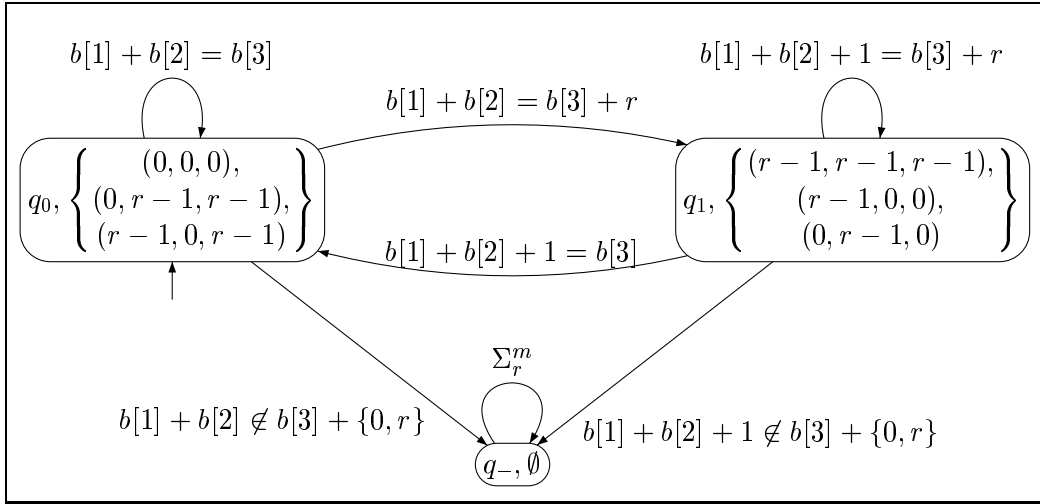


Figure 1: DVA  $\mathcal{A}_X$  representing  $X = \{x \in \mathbb{Z}^3; x[1] + x[2] = x[3]\}$

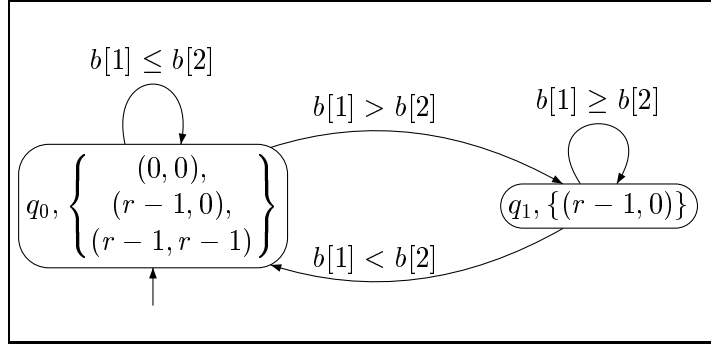


Figure 2: DVA  $\mathcal{A}_X$  representing  $X = \{x \in \mathbb{Z}^2; x[1] \leq x[2]\}$

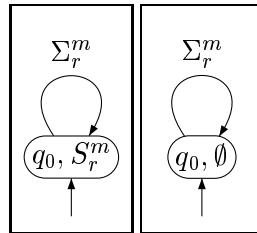


Figure 3: On the left, DVA  $\mathcal{A}_{\mathbb{Z}^m}$ . On the right, DVA  $\mathcal{A}_{\emptyset}$

As usual, function  $\delta$  is uniquely *extended* over  $Q \times (\Sigma_r^m)^*$  by  $\delta(q, \sigma_1.\sigma_2) = \delta(\delta(q, \sigma_1), \sigma_2)$ . Moreover, a tuple  $(q, \sigma, q')$  such that  $q' = \delta(q, \sigma)$  is denoted by  $q \xrightarrow{\sigma} q'$  or just  $q \rightarrow q'$ , and called a *path* from  $q$  to  $q'$  labeled by  $\sigma$ . Such a state  $q'$  is said *reachable* from  $q$  (when  $q = q_0$ , we just say that  $q'$  is *reachable*).

The *language*  $\mathcal{L}(\mathcal{A})$  *recognized* by a DVA  $\mathcal{A}$  is defined by  $\mathcal{L}(\mathcal{A}) = \{(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m; (\delta(q_0, \sigma), s) \in F_0\}$ . Thanks to the condition  $(q, s) \in F_0$  if and only if  $(q', s) \in F_0$  for every  $q \xrightarrow{s} q'$ , the language  $\mathcal{L}(\mathcal{A})$  is saturated. The set  $X = \rho_r(\mathcal{L}(\mathcal{A})) \subseteq \mathbb{Z}^m$  is called the set *represented* by the DVA  $\mathcal{A}$ .

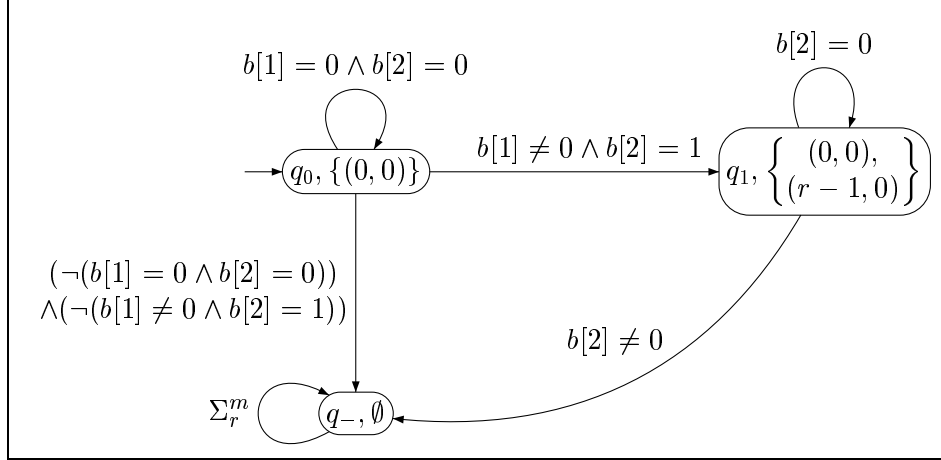


Figure 4: DVA  $\mathcal{A}_X$  representing  $X = \{x \in \mathbb{Z}^2; V_r(x[1]) = x[2]\}$

Sets represented by DVA correspond to the *r-definable sets*. Recall ([8]) that a set  $X \subseteq \mathbb{Z}^m$  is said *r-definable* if it can be defined in the first order theory  $\text{FO}(\mathbb{Z}, +, \leq, V_r)$  where  $V_r : \mathbb{Z} \rightarrow \mathbb{Z}$  is the *r*-valuation function defined by  $V_r(0) = 0$  and  $V_r(x)$  is the greatest power of  $r$  that divides  $x \in \mathbb{Z} \setminus \{0\}$  (figure 4). Recall also that a *Number Decision Diagram (NDD)* [6, 24] that represents a set  $X \subseteq \mathbb{Z}^m$ , is an automaton over  $\Sigma_r^m$  that recognizes the language  $\{\sigma.s; (\sigma, s) \in \rho_r^{-1}(X)\}$ . We do not consider NDD in this paper because the automaton obtained from a NDD by *replacing the initial state* by an other state is not a NDD in general (it does not recognizes a language of the form  $\{\sigma.s; (\sigma, s) \in \rho_r^{-1}(X')\}$  where  $X' \subseteq \mathbb{Z}^m$ ). However, DVA and NDD have slightly the same structure and we can easily compute a NDD from a DVA and conversely, that represents the same set  $X$ . In particular, we directly deduce from [8] and this remark, the following corollary 1

**Corollary 1** *A set  $X \subseteq \mathbb{Z}^m$  can be represented by a DVA if and only if it is r-definable.*

**Remark 1** *As in the NDD case, DVA can be efficiently manipulated by representing the set  $\{b \in \Sigma_r^m; q \xrightarrow{b} q'\}$  and  $\{s \in S_r^m; (q, s) \in F_0\}$  by some Binary Decision Diagrams (BDD) [9] over the alphabet  $\Sigma_r$  (and not the exponential one  $\Sigma_r^m$ ).*

## 2.2 Moving the initial state

The DVA obtained from a DVA  $\mathcal{A}$  by replacing the initial state  $q_0$  by a state  $q \in Q$  is denoted by  $\mathcal{A}_q$ . To simplify notations, when a set  $X \subseteq \mathbb{Z}^m$  is implicitly represented by a DVA  $\mathcal{A}$ , we denote by  $X_q \subseteq \mathbb{Z}^m$  the set represented by the DVA  $\mathcal{A}_q$ . We are going to characterize the set  $X_q$  in function of  $X$ . As an application, we show that any *r*-definable set  $X \subseteq \mathbb{Z}^m$  is represented by a *unique minimal DVA*.

**Proposition 1** *For any path  $q \xrightarrow{\sigma} q'$  in a DVA  $\mathcal{A}$  that represents a set  $X$ , we have  $X_{q'} = \gamma_{r, \sigma}^{-1}(X_q)$ .*

**Proof :** Without loss of generality, we can restrict our proof to a path  $q_0 \xrightarrow{\sigma} q$  in a DVA  $\mathcal{A}$  that represents a set  $X$ . Let us consider an integer vector  $x \in X_q$ . There exists a path  $q \xrightarrow{w} q'$  and  $s \in S_r^m$  such that  $x = \rho_r(w, s)$  and  $(q', s) \in F_0$ . We deduce that we have a path  $q_0 \xrightarrow{\sigma.w} q'$  with  $(q', s) \in F_0$ . Therefore  $\rho_r(\sigma.w, s) \in X$ . From  $\rho_r(\sigma.w, s) = \gamma_{r,\sigma}(\rho_r(w, s)) = \gamma_{r,\sigma}(x)$ , we deduce that  $x \in \gamma_{r,\sigma}^{-1}(X)$  and we have proved the inclusion  $X_q \subseteq \gamma_{r,\sigma}^{-1}(X)$ . For the converse inclusion, consider an integer vector  $x \in \gamma_{r,\sigma}^{-1}(X)$ . As  $\gamma_{r,\sigma}(x) \in X$ , there exists a path  $q_0 \xrightarrow{w} q'$  and  $s \in S_r^m$  such that  $\gamma_{r,\sigma}(x) = \rho_r(w, s)$  and  $(q, s) \in F_0$ . Moreover, as  $x \in \mathbb{Z}^m$ , there exists  $(w', s') \in (\Sigma_r^m)^* \times S_r^m$  such that  $x = \rho_r(w', s')$ . From the equality  $\gamma_{r,\sigma}(x) = \rho_r(w, s)$ , we deduce that  $\rho_r(\sigma.w', s') = \rho_r(w, s)$ . Lemma 1 shows that  $s' = s$  and there exists  $k_1, k_2 \in \mathbb{N}$  such that  $\sigma.w'.s^{k_1} = w.s^{k_2}$ . As we have a path  $q_0 \xrightarrow{w} q'$  with  $(q', s) \in F_0$  and  $\mathcal{A}$  is a DVA, we deduce that  $q'' = \delta(q', s^{k_2})$  is such that  $(q'', s) \in F_0$ . From  $\sigma.w'.s^{k_1} = w.s^{k_2}$ , we get that  $q_0 \xrightarrow{\sigma.w'.s^{k_1}} q''$ . In particular we have a path  $q \xrightarrow{w'.s^{k_1}} q''$  with  $(q'', s) \in F_0$ . We deduce that  $x = \rho_r(w'.s^{k_1}, s) \in X_q$  and we have proved  $\gamma_{r,\sigma}^{-1}(X) \subseteq X_q$ . Q.E.D

The previous proposition 1 proves in particular that the set  $Q_X = \{\gamma_{r,\sigma}^{-1}(X); \sigma \in (\Sigma_r^m)^*\}$  is finite when  $X$  is  $r$ -definable. The *minimal (for the number of states) DVA* that represents a  $r$ -definable set  $X \subseteq \mathbb{Z}^m$  can be easily characterized by introducing the DVA  $\mathcal{A}_X$  defined by the set of states  $Q_X$ , the transition function  $\delta_X$  defined by a  $\delta_X(X', b) = \gamma_{r,b}^{-1}(X')$  for any  $X' \in Q_X$ , the initial state  $q_{0,X} = X$ , the set of final states  $F_{0,X} = \{(X', s) \in Q_X \times S_r^m; \frac{s}{1-r} \in X'\}$ .

A DVA  $\mathcal{A}$  is said *minimal* if for any DVA  $\mathcal{A}'$  that represents the same set than  $\mathcal{A}$ , the number of states  $|Q|$  of  $\mathcal{A}$  is less than or equal to the number of states  $|Q'|$  of  $\mathcal{A}'$ . Two DVA  $\mathcal{A}_1 = (Q_1, \Sigma_r^m, \delta_1, q_{0,1}, F_{0,1})$  and  $\mathcal{A}_2 = (Q_2, \Sigma_r^m, \delta_2, q_{0,2}, F_{0,2})$  are said *isomorph* if there exists a *one-to-one relation*  $\sim \subseteq Q_1 \times Q_2$  such that  $\delta_1(q_1, b) \sim \delta_2(q_2, b)$  and  $\{s \in S_r^m; (q_1, s) \in F_{0,1}\} = \{s \in S_r^m; (q_2, s) \in F_{0,2}\}$  for any  $q_1 \sim q_2$ , and such that  $q_{0,1} \sim q_{0,2}$ .

**Theorem 1** *For any  $r$ -definable set  $X \subseteq \mathbb{Z}^m$ , the DVA  $\mathcal{A}_X$  is the unique (up to isomorphism) minimal DVA that represents  $X$ .*

**Proof :** First remark that  $\mathcal{A}_X$  is a DVA that represents  $X$ . Next, let us consider a minimal DVA  $\mathcal{A} = (Q, \Sigma_r^m, \delta, q_0, F_0)$  that represents  $X$ . Proposition 1 proves that there exists a function  $f : Q_X \rightarrow Q$  such that  $X_{f(X')} = X'$  for any  $X' \in Q_X$ . In particular  $|Q_X| \leq |Q|$  and as  $\mathcal{A}$  is minimal, we have  $|Q_X| = |Q|$  and in particular  $\mathcal{A}_X$  is also minimal. Moreover, we deduce that  $f$  is a one-to-one function. Just remark that  $\mathcal{A}$  and  $\mathcal{A}_X$  are isomorph for the one-to-one relation  $\sim = \{(X', f(X'));$   $X' \in Q_X\}$ . Q.E.D

From the previous theorem 1 and corollary 1, we deduce that a set  $X \subseteq \mathbb{Z}^m$  is  $r$ -definable if and only if  $Q_X = \{\gamma_{r,\sigma}^{-1}(X); \sigma \in (\Sigma_r^m)^*\}$  is finite.

### 2.3 Replacing the set of final states

Given a DVA  $\mathcal{A}$ , the class of subsets  $F \subseteq Q \times S_r^m$  such that  $(q, s) \in F$  if and only if  $(q', s) \in F$  for any transition  $q \xrightarrow{s} q'$ , is denoted by  $\mathcal{F}_{\mathcal{A}}$ . The DVA obtained from a DVA  $\mathcal{A}$  by replacing the set of final states  $F_0$  by a set  $F \in \mathcal{F}_{\mathcal{A}}$  is denoted by  $\mathcal{A}^F$ . To simplify notions, when a set  $X \subseteq \mathbb{Z}^m$  is implicitly represented by a DVA  $\mathcal{A}$ , we denote by  $X^F$  the set represented by the DVA  $\mathcal{A}^F$ . In this section, the set  $\mathcal{F}_{\mathcal{A}}$  is geometrically characterized by introducing the notion of *eyes*, *semi-eyes* and *kernel*.

Let us consider the *equivalence relation*  $\sim_{\mathcal{A}}$  over  $Q \times S_r^m$  defined by  $(q_1, s_1) \sim_{\mathcal{A}} (q_2, s_2)$  if and only if  $s_1 = s_2$  and  $\delta(q_1, s_1^*) \cap \delta(q_2, s_2^*) \neq \emptyset$ .

An *eye*  $Y$  is an *equivalence class* for the relation  $\sim_{\mathcal{A}}$  (see figure 5). A *semi-eye* is a finite union of eyes. Remark that the class of semi-eyes is exactly  $\mathcal{F}_{\mathcal{A}}$ .

Let us consider the function  $\delta_e : Q \times S_r^m \rightarrow Q \times S_r^m$  defined by  $\delta_e(q, s) = (\delta(q, s), s)$ .

The *kernel*  $\ker(Y)$  of a subset  $Y \subseteq Q \times S_r^m$  is defined as  $\ker(Y) = \bigcap_{n \in \mathbb{N}} \delta_e^n(Y)$  and corresponds to the greatest (for  $\subseteq$ ) fix-point for  $\delta_e$  included in  $Y$ . Remark that the kernel of any eye  $Y$  is a non

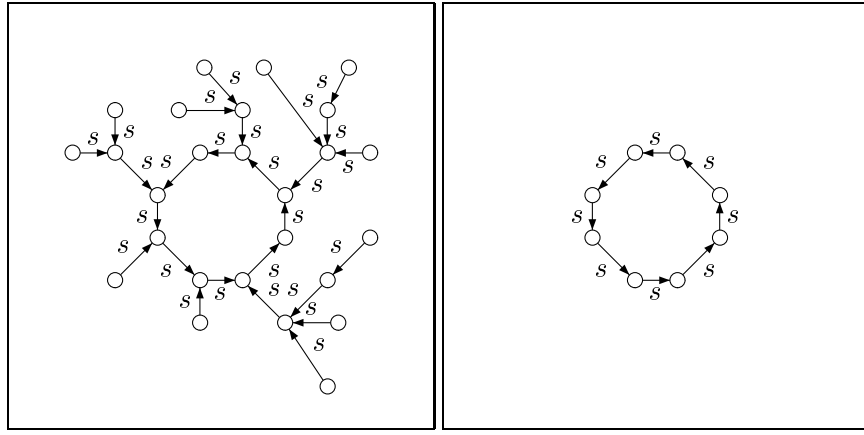


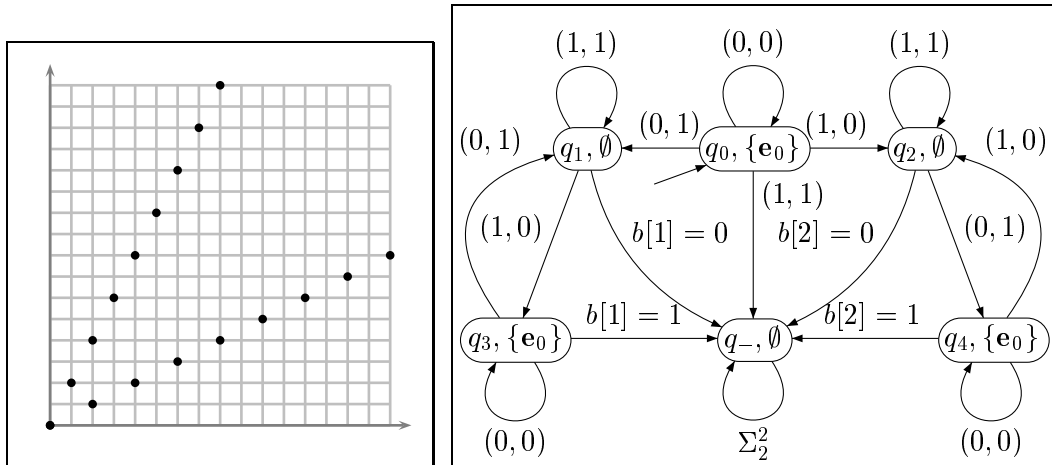
Figure 5: On the left an eye. On the right its kernel.

empty set of the form  $\ker(Y) = \{(q_0, s), \dots, (q_{n-1}, s), (q_n, s) = (q_0, s)\}$  such that  $\delta(q_i, s) = q_{i+1}$  for any  $i \in \{0, \dots, n-1\}$  (see figure 5).

**Example 1** Let  $\mathcal{A}_X$  be the minimal DVA representing  $X = \{x \in \mathbb{Z}^3; x[1] + x[2] = x[3]\}$  given in figure 1. The eyes of  $\mathcal{A}$  are  $\{(q_0, (0, 0))\}$ ,  $\{(q_0, (r-1, r-1))\}$ ,  $\{(q_1, (0, 0))\}$ ,  $\{(q_1, (r-1, r-1))\}$ ,  $\{(q_0, (0, r-1)), (q_1, (0, r-1))\}$ , and  $\{(q_0, (r-1, 0)), (q_1, (r-1, 0))\}$ .

### 3 Presburger-definable DVA

A subset  $X \subseteq \mathbb{Z}^m$  is said *Presburger-definable* if it can be defined by a formula in the first order theory  $\text{FO}(\mathbb{Z}, +, \leq)$  (see figure 6). A DVA  $\mathcal{A}$  is said *Presburger-definable* if the set represented by  $\mathcal{A}$  is Presburger-definable. A set  $X$  is said *structurally Presburger-definable* if the minimal DVA  $\mathcal{A}$  that represents  $X$ , is such that  $\mathcal{A}_q^F$  is Presburger-definable for any state  $q \in Q$  and for any semi-eyes  $F \in \mathcal{F}_{\mathcal{A}}$ . Naturally, as  $\mathcal{A}_{q_0}^{F_0}$  represents  $X$ , a structurally Presburger-definable set is Presburger-definable. In this section, we prove the converse.

Figure 6: A Presburger-definable set  $\{x \in \mathbb{N}^2; (x[1] = 2 \cdot x[2]) \vee (2 \cdot x[1] = x[2])\}$  and its minimal DVA  $\mathcal{A}_X$  in basis  $r = 2$ .

**Remark 2** A linear set  $X$  of  $\mathbb{Z}^m$  is a set of the form  $X = b + \sum_{p \in P} \mathbb{N} \cdot p$  where  $b \in \mathbb{Z}^m$  is called the basis and  $P \subseteq \mathbb{Z}^m$  is a finite subset of  $\mathbb{Z}^m$  called the set of periods. A semi-linear set of  $\mathbb{Z}^m$  is a finite union of linear sets of  $\mathbb{Z}^m$ . Recall that a set  $X$  is Presburger-definable if and only if it is semi-linear [12].

**Example 2** The Presburger-definable set  $X = \{x \in \mathbb{N}^2; (x[1] = 2.x[2]) \vee (2.x[1] = x[2])\}$  and its minimal DVA  $\mathcal{A}_X$  in basis  $r = 2$  are given in figure 6. Remark that the set of final states  $F_0$  can be decomposed into 3 eyes  $Y_0 = \{(q_0, \mathbf{e}_0)\}$ ,  $Y_3 = \{(q_3, \mathbf{e}_0)\}$  and  $Y_4 = \{(q_4, \mathbf{e}_0)\}$ . The DVA  $\mathcal{A}_X^{Y_0}$ ,  $\mathcal{A}_X^{Y_3}$  and  $\mathcal{A}_X^{Y_4}$  respectively represent  $X^{Y_0} = \{\mathbf{e}_0\}$ ,  $X^{Y_3} = \{x \in \mathbb{N}^2 \setminus \{\mathbf{e}_0\}; x[1] = 2.x[2]\}$  and  $X^{Y_4} = \{x \in \mathbb{N}^2 \setminus \{\mathbf{e}_0\}; 2.x[1] = x[2]\}$ .

From proposition 1, we get the following corollary.

**Corollary 2** For any reachable state  $q$  of a Presburger-definable DVA  $\mathcal{A}$ , the DVA  $\mathcal{A}_q$  is Presburger-definable.

**Proof :** Let  $\mathcal{A}$  be a DVA that represents a Presburger-definable set  $X$  and consider a reachable state  $q$  of  $\mathcal{A}$ . There exists a path  $q_0 \xrightarrow{\sigma} q$ . Proposition 1 proves that  $X_q = \gamma_{r,\sigma}^{-1}(X)$ . As  $X$  is Presburger-definable, there exists a Presburger-formula  $\phi$  that defines  $X$ . Now, just remark that  $X_q$  is defined by the Presburger formula  $\phi_\sigma(x) := \exists x' (x' = r^{|\sigma|}.x + \gamma_{r,\sigma}(\mathbf{e}_0) \wedge \phi(x'))$ . Hence  $\mathcal{A}_q$  is Presburger-definable. Q.E.D

A quantification elimination shows that a Presburger-definable set  $X$  is a boolean combination in  $\mathbb{Z}^m$  of sets of the form  $X = \{x \in \mathbb{Z}^m; x[i] \in c + n.\mathbb{Z}\}$  where  $(i, c, n) \in \{1, \dots, m\} \times \mathbb{Z} \times (\mathbb{N} \setminus \{0\})$ , and sets of the form  $X = \{x \in \mathbb{Z}^m; \langle \alpha, x \rangle \leq c\}$  where  $(\alpha, c) \in (\mathbb{Z}^m \setminus \{0\}) \times \mathbb{Z}$ . The following technical lemmas 2 and 3 prove that these sets are structurally Presburger-definable.

**Lemma 2** The set  $X = \{x \in \mathbb{Z}^m; x[i] \in c + n.\mathbb{Z}\}$  where  $(i, c, n) \in \{1, \dots, m\} \times \mathbb{Z} \times (\mathbb{N} \setminus \{0\})$  is structurally Presburger-definable.

**Proof :** Let  $\mathcal{A}$  be the minimal DVA that represents  $X = \{x \in \mathbb{Z}^m; x[i] \in c + n.\mathbb{Z}\}$ . There exists a unique integer  $k \in \mathbb{N}$  such that  $n_0 = \frac{n}{r^k}$  is a  $r$ -prime integer (an integer relatively prime with  $r$ ). Let us consider the set  $\mathcal{L}$  of words  $\sigma \in (\Sigma_r^m)^k$  such that  $\gamma_{r,\sigma}^{-1}(X) \neq \emptyset$ . Remark that for any word  $\sigma \in \mathcal{L}$ , we have  $\gamma_{r,\sigma}^{-1}(X) = \{x \in \mathbb{Z}^m; r^k.x[i] \in c - \gamma_{r,\sigma}(\mathbf{e}_0)[i] + n.\mathbb{Z}\}$ . As  $\gamma_{r,\sigma}^{-1}(X) \neq \emptyset$ , we deduce that  $c_\sigma = \frac{c - \gamma_{r,\sigma}(\mathbf{e}_0)[i]}{r^k}$  is an integer, and in particular we get  $\gamma_{r,\sigma}^{-1}(X) = \{x \in \mathbb{Z}^m; x[i] \in c_\sigma + n_0.\mathbb{Z}\}$ . As  $n_0$  is  $r$ -prime, there exists an integer  $k_0 \in \mathbb{N}$  such that  $r^{k_0} \in 1 + n_0.\mathbb{Z}$ . For any  $\sigma \in \mathcal{L}$  and for any  $(w, s) \in ((\Sigma_r^m)^{k_0})^* \times S_r^m$ , we have:

$$\begin{aligned} \gamma_{r,\sigma,w}^{-1}(X) &= \gamma_{r,w}^{-1}(\{x \in \mathbb{Z}^m; x[i] \in c_\sigma + n_0.\mathbb{Z}\}) \\ &= \{x \in \mathbb{Z}^m; r^{|w|}.x[i] + \gamma_{r,w}(\mathbf{e}_0)[i] \in c_\sigma + n_0.\mathbb{Z}\} \\ &= \{x \in \mathbb{Z}^m; x[i] \in c_\sigma + \frac{s}{1-r} - \rho_r(w, s)[i] + n_0.\mathbb{Z}\} \end{aligned}$$

Let us consider an eye  $Y$  of  $\mathcal{A}$ , let  $s \in S_r^m$  be the unique sign vector such that  $Y \subseteq Q \times \{s\}$ . Let us consider the Presburger-definable set  $Z_s = \{\rho_r(\sigma, s); \sigma \in (\Sigma_r^m)^*\}$  of vectors with the same sign  $s$ .

We first assume that  $X_q \neq \emptyset$  for any  $(q, s) \in \ker(Y)$ . We denote by  $P$  the set of  $p \in \mathbb{Z}$  such that  $\{x \in \mathbb{Z}^m; x[i] \in -p + n_0.\mathbb{Z}\} \in \{X_q; (q, s) \in \ker(Y)\}$ . Remark that  $P$  is Presburger-definable because

$P = (P \cap \{0, \dots, n_0 - 1\}) + n_0 \cdot \mathbb{Z}$ . Moreover, we have:

$$\begin{aligned}
x \in X^Y &\iff \exists \sigma \in (\Sigma_r^m)^* \ x = \rho_r(\sigma, s) \wedge (\delta(q_0, \sigma), s) \in Y \\
&\iff \exists \sigma \in \mathcal{L} \ \exists w \in ((\Sigma_r^m)^{k_0})^* \ x = \rho_r(\sigma.w, s) \wedge (\delta(q_0, \sigma.w), s) \in \ker(Y) \\
&\iff \exists \sigma \in \mathcal{L} \ \exists w \in ((\Sigma_r^m)^{k_0})^* \ \begin{cases} x = \gamma_{r,\sigma}(\rho_r(w, s)) \\ \wedge \rho_r(w, s)[i] \in c_\sigma + \frac{s}{1-r} + P \end{cases} \\
&\iff \exists \sigma \in \mathcal{L} \ \exists z \in Z_s \ x = \gamma_{r,\sigma}(z) \wedge z[i] \in c_\sigma + \frac{s}{1-r} + P
\end{aligned}$$

We have proved that  $X^Y$  is Presburger-definable.

Finally, assume that  $X_q = \emptyset$  for at least one  $(q, s) \in \ker(Y)$ . We have  $X^Y = Z_s \setminus \bigcup_{Y' \in \mathcal{C} \setminus \{Y\}} X^{Y'}$  where  $\mathcal{C}$  is the set of eyes  $Y' \subseteq Q \times \{s\}$ . Remark that if there exists an eye  $Y' \in \mathcal{C} \setminus \{Y\}$  and  $(q', s) \in \ker(Y')$  such that  $X_{q'} = \emptyset$ , as  $\mathcal{A}$  is minimal, we get  $q = q'$  and in particular  $Y = Y'$  which is impossible. From the previous paragraph, we deduce that  $X^{Y'}$  is Presburger-definable for any  $Y' \in \mathcal{C} \setminus \{Y\}$ . Therefore  $X^Y$  is Presburger-definable. Q.E.D

**Lemma 3** *The set  $X = \{x \in \mathbb{Z}^m; \langle \alpha, x \rangle \leq c\}$  where  $(\alpha, c) \in (\mathbb{Z}^m \setminus \{0\}) \times \mathbb{Z}$  is structurally Presburger-definable.*

**Proof :** Let  $\mathcal{A}$  be the minimal DVA that represents  $X = \{x \in \mathbb{Z}^m; \langle \alpha, x \rangle \leq c\}$ . For any  $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$ , and for any  $k \in \mathbb{N}$ , we have:

$$\gamma_{r,\sigma,s^k}^{-1}(X) = \left\{ x \in \mathbb{Z}^m; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \leq \frac{c - \langle \alpha, \rho_r(\sigma, s) \rangle}{r^{|\sigma|+k}} \right\}$$

In particular, for any  $(\sigma, s) \in (\Sigma_r^m)^* \times S_r^m$ , there exists  $k_0 \in \mathbb{N}$  such that for any integer  $k \geq k_0$ , we have:

$$\gamma_{r,\sigma,s^k}^{-1}(X) = \begin{cases} \{x \in \mathbb{Z}^m; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \leq 0\} & \text{if } \langle \alpha, \rho_r(\sigma, s) \rangle \leq c \\ \{x \in \mathbb{Z}^m; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle < 0\} & \text{if } \langle \alpha, \rho_r(\sigma, s) \rangle > c \end{cases}$$

Let us consider an eye  $Y$  and the unique sign digit vector  $s \in S_r^m$  such that  $Y \subseteq Q \times \{s\}$ . Let us consider the Presburger-definable set  $Z_s = \{\rho_r(\sigma, s); \sigma \in (\Sigma_r^m)^*\}$  of vectors with the same sign  $s$ .

From the previous equality, we deduce that there exists  $\# \in \{<, \leq\}$  such that for any  $(q, s) \in \ker(Y)$  we have  $X_q = \{x \in \mathbb{Z}^m; \left\langle \alpha, x - \frac{s}{1-r} \right\rangle \# 0\}$ . In particular  $\ker(Y)$  is reduced to  $\ker(Y) = \{(q, s)\}$ . Let us consider  $\#' \in \{\leq, >\}$  such that  $(\#, \#') \in \{(\leq, \leq), (<, >)\}$ . We have:

$$\begin{aligned}
x \in X^Y &\iff \exists \sigma \in (\Sigma_r^m)^* \ (\delta(q, \sigma.s^*), s) \cap \ker(Y) \neq \emptyset \\
&\iff x \in Z_s \wedge \langle \alpha, x \rangle \#' c
\end{aligned}$$

Therefore  $X^Y$  is Presburger-definable. Q.E.D

**Theorem 2** *A set  $X$  is structurally Presburger-definable if and only if it is Presburger-definable.*

**Proof :** Recall that a quantification elimination shows that a Presburger-definable set is a boolean combination in  $\mathbb{Z}^m$  of sets of the form  $X = \{x \in \mathbb{Z}^m; x[i] \in c + n \cdot \mathbb{Z}\}$  and sets of the form  $X = \{x \in \mathbb{Z}^m; \langle \alpha, x \rangle \leq c\}$ . Lemmas 2 and 3 prove that these sets are structurally Presburger-definable. Moreover, as the complement of a structurally Presburger-definable set remains structurally Presburger-definable, it is sufficient to prove that the intersection  $X = X_1 \cap X_2$  of two structurally Presburger-definable sets  $X_1$  and  $X_2$  remains structurally Presburger definable. Let  $\mathcal{A}_1, \mathcal{A}_2$

and  $\mathcal{A}'$  be the minimal DVA that represent respectively  $X_1$ ,  $X_2$  and  $X$ . Remark that  $X$  is represented by the *Cartesian product*  $\mathcal{A} = (Q_1 \times Q_2, \Sigma_r^m, \delta, q_0, F_0)$  where  $\delta((q_1, q_2), b) = (\delta_1(q_1, b), \delta_2(q_2, b))$ ,  $q_0 = (q_{1,0}, q_{2,0})$ , and  $F_0 = F_{1,0} \times F_{2,0}$ . Remark that for any eye  $Y$  of the DVA  $\mathcal{A}'$ , there exists a finite sequence  $(Y_{1,i}, Y_{2,i})_{i \in I}$  where  $Y_{1,i}$  and  $Y_{2,i}$  are some eyes of respectively  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , such that  $X^Y$  is represented by the DVA  $\mathcal{A}^{\bigcup_{i \in I} Y_{1,i} \times Y_{2,i}}$ . Therefore  $X^Y = \bigcup_{i \in I} X_1^{Y_{1,i}} \cap X_2^{Y_{2,i}}$  is Presburger-definable. In particular  $X$  is structurally Presburger-definable. We are done. Q.E.D

## 4 Future work

We have proved that any Presburger-definable set is structurally Presburger-definable. In particular, the widening operator for DVA introduced by Bartzis and Bultan provides Presburger-definable DVA from the widening of two Presburger-definable DVA. We are interested in extending the geometrical widening operators known for the *closed convex polyhedrons* [10], to the Presburger-definable DVA.

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